トポロジカル秩序とクォーク閉じ込め問題

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@ 九州，Dec.25,‘08

- Part 1

What is topological order ?

Fractionalization and topological orders.

MS, M. Kohmoto and Y.-S.Wu, Phys. Rev. Lett. 97, 010601 (2006)

- Part 2

Quark (De)confinement as a Topological Order

MS, Phys. Rev. D77, 045013 (2008) + alpha

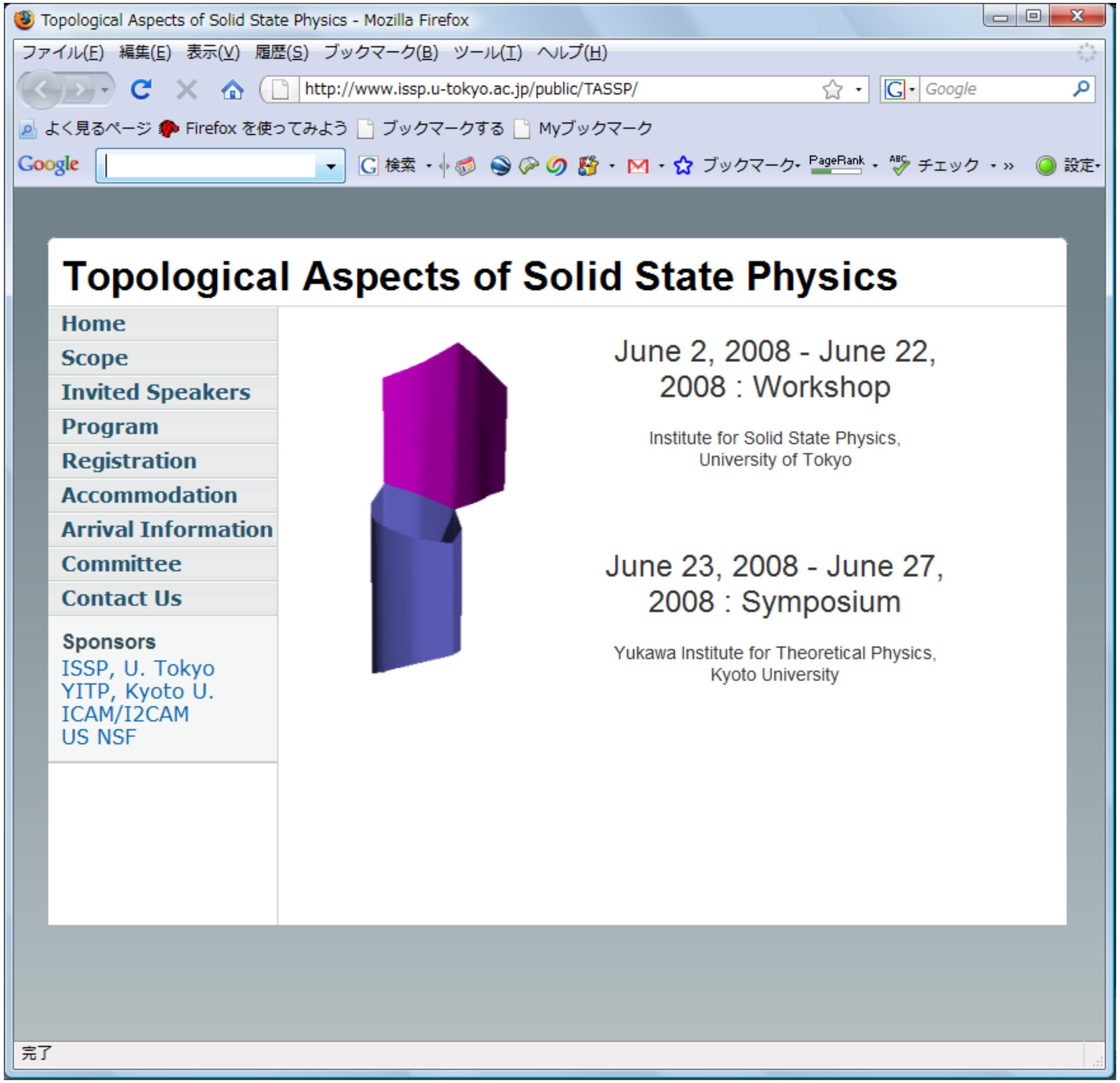
Physics of topological orders

- Bulk-edge correspondence ・・・ holographic relation
  index theorem

- fractionalization ・・・ fractional charge
  fractional statistics
  (anyon, non-abelion)
  fractional QHE

- topology-changing phase transition

- (topological) entanglement entropy ・・・quantum computer



http://www.issp.u-tokyo.ac.jp/public/TASSP

① What is topological order ?

Orders: conventionally understood in terms of SSB

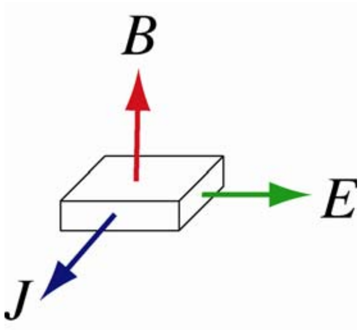
spontaneous symmetry breaking

$$\langle \phi \rangle \neq 0$$

But, recently it is recognized that there exist orders which cannot be described by spontaneous symmetry breaking

Example

fractional quantum Hall system

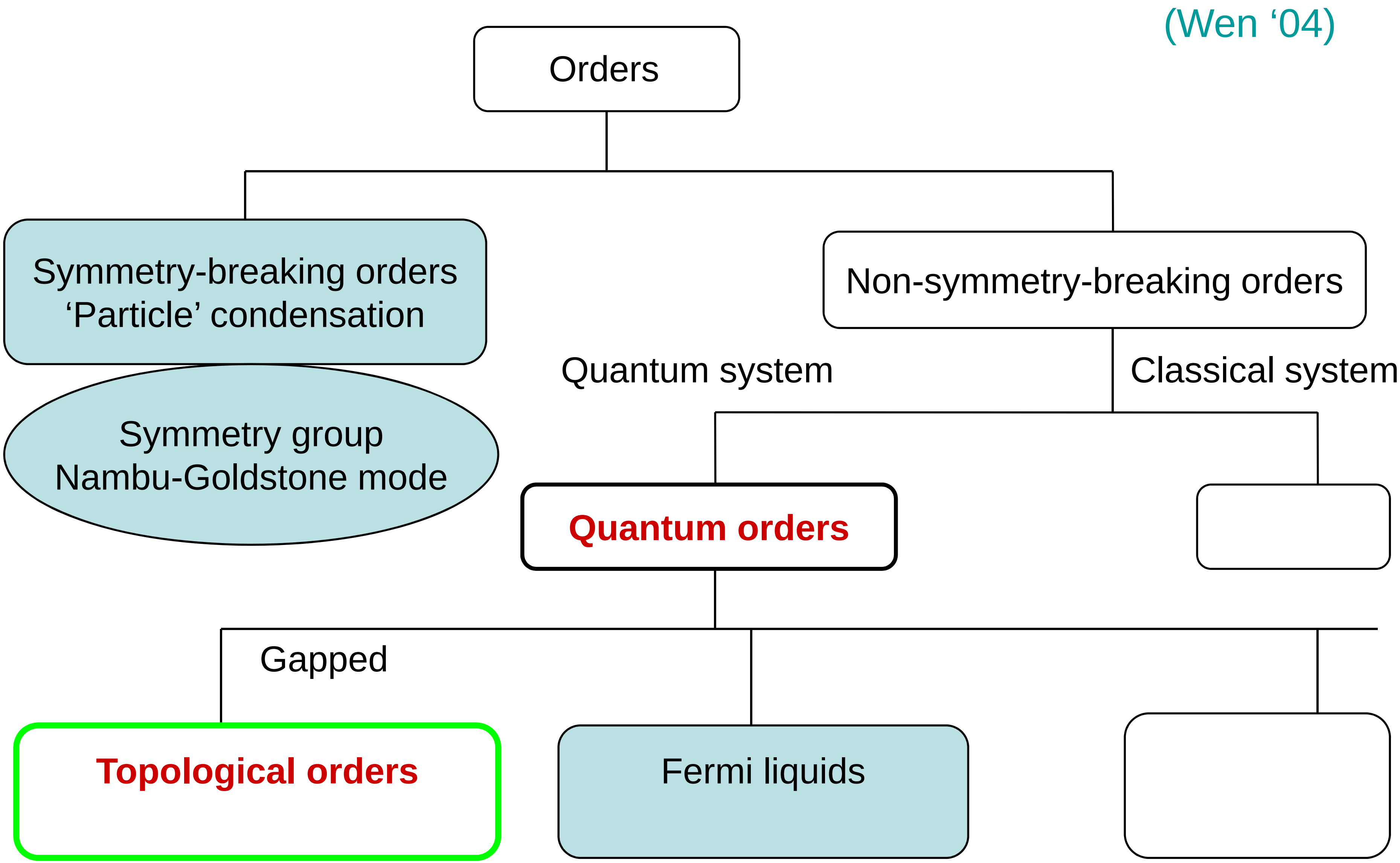


$$\sigma_{xy} \equiv \frac{J}{E} = \frac{e^2}{h}\nu$$

$$\nu = 2\pi n_e / eB$$

R.Willett et al. '87

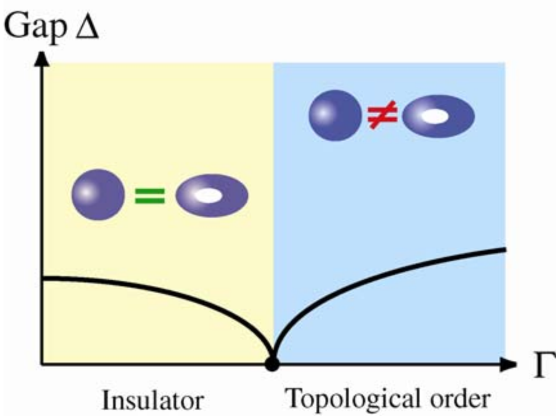
- Phase transitions (jumps of Hall conductance) occur by changing the magnetic field
- The FQH states do not break the symmetry of the Hamiltonian (quantum liquid state).
- They have excitations with different fractional charges. $e^* = e\nu$


(Wen '04)
Orders
Symmetry-breaking orders
'Particle' condensation
Non-symmetry-breaking orders
Symmetry group
Nambu-Goldstone mode
Quantum system
Classical system
Quantum orders
Gapped
Topological orders
Fermi liquids

Topological order

◆ Topological orders are conventionally characterized by the ground-state degeneracy depending on the topology of the space (topological degeneracy)

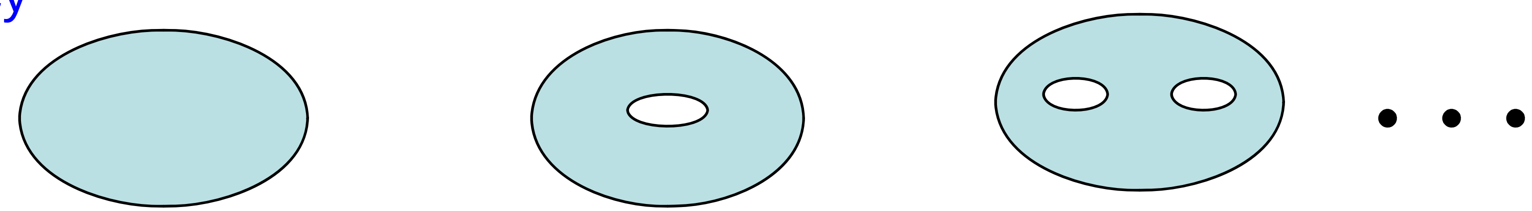
Wen '90



fractional quantum Hall systems

ex.) Laughlin state $\nu = \frac{1}{q}$

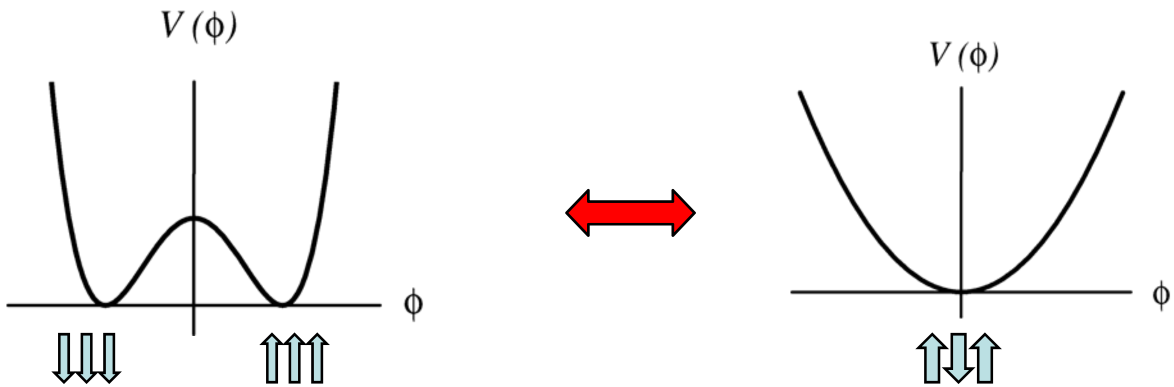
degeneracy



1 q q^2

Wen-Niu '90

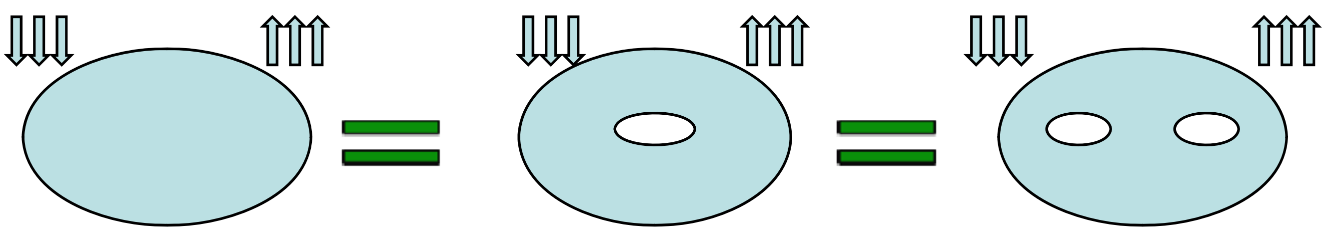
The ground state degeneracy is useful even for symmetry breaking orders



ordered
2-fold degeneracy

not ordered
no degeneracy

But, the degeneracy is **independent of the topology !**



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At present, several different systems are known to exhibit topological orders.

- ◆ boson system
- ◆ fermion system
- ◆ in the presence or in the absence of magnetic filed (without or with time-reversal invariance)
- ◆ 2+1 and 3+1 dim. system

- ■ fractional quantum Hall systems
- ■ insulators
- ■ (frustrated) spin systems
- ■ unconventional superconductors Sr_2RuO_4
- ■ superfluid ^{3}He A-phase

X.G.Wen ‘91,
Read-Sachdev ‘91,
Senthil-Fisher ‘00,
Moessner-Sondhi ‘01,
Misguichi-Serban-Pesquir ‘02,
Balents-Fisher-Girvin ‘02,
Motrunich-Senthil ‘02,
Kitaev, ‘97,
…….

Common characteristics

① All the known models have an excitation with **fractional charge**.

Oshikawa-Senthil ‘06

In addition, some models have the following other fractionalization

② Some models have an excitation with **fractional statistics** or **non-abelian statistics**.
③ Some models show **fractional quantum Hall effects**.

All these fractionalizations induce hidden symmery (topological discrete algebra).

Fractionalization = Topological order

Plan of part1

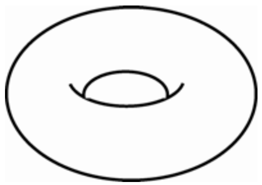
① Topological discrete algebra
② Ground state degeneracy
③ Fractionalization and topological invariant
(Fractional Quantum Hall effect)

② Topological Discrete Algebra (d=2+1)

Our assumptions are the following

definition of charge

① The system is on a torus.
② There exists an exact U(1) symmetry.
③ The system is gapped.
④ Charge fractionalization occurs.



In other words, we assume that
there exist a quasi-particle with fractional charge

$$e^* = \frac{p}{q} e$$

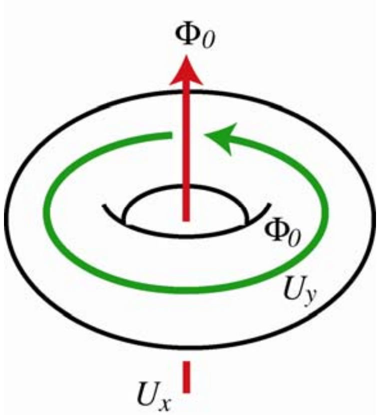
p, q coprime integers

(e charge of the constituent particle)

We start with the following non-trivial processes on a torus

(Wu-Hatsugai-Kohmoto '91)

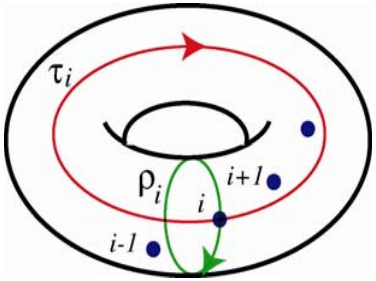
a. adiabatic unit flux insertions through holes of torus



$$U_x,\ U_y$$

$$\Phi_0 = \frac{2\pi}{e}$$ a unit flux

b. translations of i-th quasi-particle along loops of torus



$$\rho_i,\ \tau_i$$

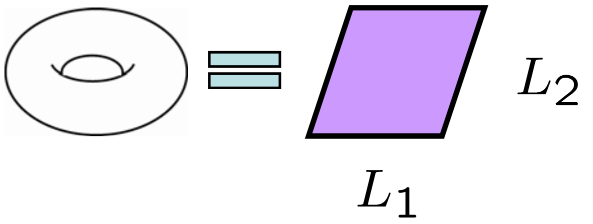
◆Note that the unit flux insertions are unitary equivalent to the following adiabatic changes of the boundary conditions.

Consider the twisted boundary condition on the torus

phase

$$\psi_n(x_i + L_1) = e^{i\theta_x}\psi_n(x_i)$$
$$\psi_n(y_i + L_2) = e^{i\theta_y}\psi_n(y_i)$$

phase



By large gauge transformation

$$U(\Phi_x, \Phi_y) = e^{i(e\Phi_x/L_1)(x_1+\cdots+x_N)+i(e\Phi_y/L_2)(y_1+\cdots+y_N)}$$

1. the flux is reduced as $$-i\frac{\partial}{\partial x_i} \to -i\frac{\partial}{\partial x_i} - e\frac{\Phi_x}{L_1}$$
2. the boundary conditions are changed as

$$U(\Phi_x, \Phi_y, x_i+L_1)\psi(x_i+L_1) = e^{i(e\Phi_x+\theta_x)}U(\Phi_x, \Phi_y, x_i)\psi(x_i)$$

Using the large gauge transformation, we can delete the inserted unit flux completely, but the boundary condition parameters θ_x and θ_y change by one period.

◆ Thus U_x (U_y) is unitary equivalent to adiabatic change of θ_x (θ_y) by 2π

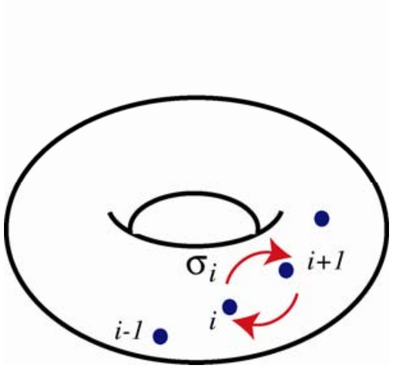
$$\psi_n(x_i + L_1) = e^{i\theta_x}\psi_n(x_i)$$
$$\psi_n(y_i + L_2) = e^{i\theta_y}\psi_n(y_i)$$

$$\theta_x \xrightarrow{U_x} \theta_x + 2\pi$$
$$\theta_y \xrightarrow{U_y} \theta_y + 2\pi$$

The spectrum should be invariant under U_x and U_y
("gauge invariance due to large gauge transformation")

We also consider the exchange of quasi-particles.

c. exchange between i-th and (i+1)-th quasi-particles



σ_i

The reason why we take into account this operation is that the translation along the loops are not independent to the exchange of quasi-particles. Indeed they form the braid group algebra.

Braid Group on torus

$$A_{j,i} \equiv \tau_j^{-1} \rho_i \tau_j \rho_i^{-1}, \quad C_{j,i} \equiv \rho_j^{-1} \tau_i \rho_j \tau_i^{-1}, \quad (1 \le i < j \le N),$$

$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad (1 \le k \le N-3,\ |l-k| \ge 2),$$
$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad (1 \le k \le N-2),$$
$$\underline{\tau_{i+1} = \sigma_i^{-1} \tau_i \sigma_i^{-1}}, \quad \underline{\rho_{i+1} = \sigma_i \rho_i \sigma_i,}$$
$$\tau_1 \sigma_j = \sigma_j \tau_1, \quad \rho_1 \sigma_j = \sigma_j \rho_1, \quad \underline{\sigma_i^2 = A_{i+1,i},}$$
$$(1 \le i \le N-1,\ 2 \le j \le N-1)$$

$$A_{m,l} \tau_k = \tau_k A_{m,l}, \quad A_{m,l} \rho_k = \rho_k A_{m,l},$$
$$\tau_i \tau_j = \tau_j \tau_i, \quad \rho_i \rho_j = \rho_j \rho_i,$$
$$C_{j,i} = (\tau_i \tau_j) A_{j,i}^{-1} (\tau_j^{-1} \tau_i^{-1}), \quad A_{j,i} = (\rho_i \rho_j) C_{j,i}^{-1} (\rho_j^{-1} \rho_i^{-1}),$$
$$C_{j,i} = (A_{j,j-1}^{-1} \cdots A_{j,i+1}^{-1}) A_{j,i}^{-1} (A_{j,i+1} \cdots A_{j,j-1}),$$
$$\tau_1 \rho_1 \tau_1^{-1} \rho_1^{-1} = A_{2,1} A_{3,1} \cdots A_{N-1,1} A_{N,1}$$
$$(1 \le k < l < m \le N,\ 1 \le i < j \le N)$$

The commutation relations between the braid group operators and the flux insertions are determined by AB effect.

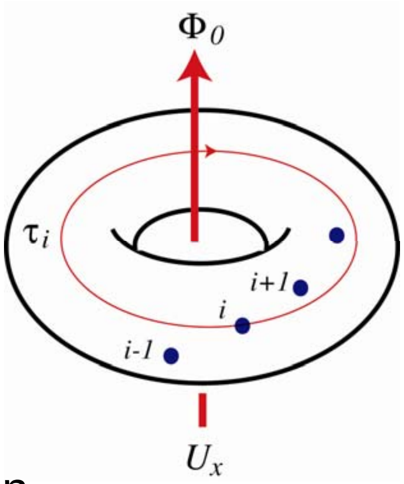
From AB effect, we have

$$U_x \tau_i = e^{-2\pi i p/q} \tau_i U_x$$

AB phase

translation after flux insertion

$$e^* = \frac{p}{q} e$$



In a similar manner, we obtain

$$U_y \rho_i = e^{-2\pi i p/q} \rho_i U_y$$

$$U_x \rho_i = \rho_i U_x, \quad U_x \sigma_i = \sigma_i U_x,$$
$$U_y \tau_i = \tau_i U_y, \quad U_y \sigma_i = \sigma_i U_y.$$

On the other hand, the commutation relation between the flux insertion operators is determined by Schur's lemma

$U_x U_y U_x^{-1} U_y^{-1}$ commutes with all the braid group operators

Schur's lemma

⇒

$$U_x U_y U_x^{-1} U_y^{-1} = const = e^{2\pi i \lambda}$$

To have a finite dimensional representation, we have

$$\lambda = \frac{k}{l} \qquad k, l \text{ coprime integers}$$

Thus, our tool to examine the topological order is the following.

① Braid Group

② AB effect

$$U_x \tau_i = e^{-2\pi i p/q} \tau_i U_x, \quad U_y \rho_i = e^{-2\pi i p/q} \rho_i U_y$$

$$U_x \rho_i = \rho_i U_x, \quad U_x \sigma_i = \sigma_i U_x,$$

$$U_y \tau_i = \tau_i U_y, \quad U_y \sigma_i = \sigma_i U_y.$$

③ Schur's lemma

$$U_x U_y = e^{2\pi i \lambda} U_y U_x, \quad \lambda = k/l$$

Topological discrete algebra (1)

If the quasi-particle obeys **abelian statistics**, the algebra is simplified.

$$\sigma_i = e^{i\theta}\mathbf{1}, \text{ (boson, fermion, anyon)}$$

The solution of braid group is

$$\tau_j = e^{-2i\theta(j-1)}T_x, \quad \rho_j = e^{2i\theta(j-1)}T_y$$
$$T_xT_y = e^{-2i\theta}T_yT_x, \quad \theta = \frac{m}{n}\pi$$

m, n coprime integers

$$T_xT_y = e^{-2\pi i m/n}T_yT_x, \quad U_xU_y = e^{2\pi i\lambda}U_yU_x,$$
$$U_xT_xU_x^{-1} = e^{-2\pi i p/q}T_x, \quad U_yT_xU_y^{-1} = T_x,$$
$$U_xT_yU_x^{-1} = T_y, \quad U_yT_yU_y^{-1} = e^{-2\pi i p/q}T_y$$

Topological discrete algebra (2)

$$\theta = \pi\frac{m}{n}, \ e^* = \frac{p}{q}e, \ \lambda = \frac{k}{l}$$

The statistical property is naturally combined with the charge fractionalization in terms of the topological algebra.

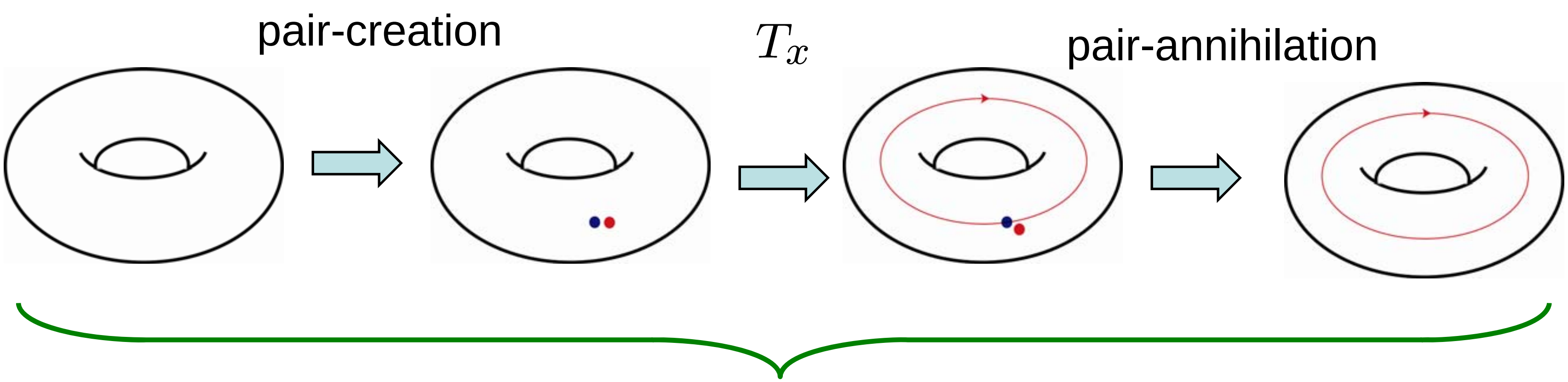
Questions:

1. Can the topological algebra explain the ground state degeneracy ?

2. (What is the physical meaning of the parameter λ ?)

③ Ground state degeneracy

To count ground state degeneracy, we define T_x on the ground state



definition of T_x on the ground state

In a similar manner, we define T_y on the ground state

Let us take the basis of the ground state to be an eigenstate of T_x

$$T_x|\eta\rangle = e^{i\eta}|\eta\rangle$$

then, we have n different eigenstates

$$\Longrightarrow \quad T_x(T_y^s|\eta\rangle) = e^{i(\eta - 2\pi sm/n)}T_y^s|\eta\rangle \qquad s = 1, \cdots, n$$

n-fold degeneracy

We can also take the basis to be an eigenstate of T_x and T_y^n

$$T_x T_y^n = T_y^n T_x$$

$$T_x|\eta_1, \eta_2\rangle = e^{i\eta_1}|\eta_1, \eta_2\rangle, \quad T_y^n|\eta_1, \eta_2\rangle = e^{i\eta_2}|\eta_1, \eta_2\rangle$$

In this case, we have qQ different eigenstates

$$\Rightarrow \quad T_x(U_x^s U_y^t|\eta_1, \eta_2\rangle) = e^{i(\eta_1 + 2\pi sp/q)} U_x^s U_y^t|\eta_1, \eta_2\rangle$$
$$T_y^n(U_x^s U_y^t|\eta_1, \eta_2\rangle) = e^{i(\eta_2 + 2\pi tnp/q)} U_x^s U_y^t|\eta_1, \eta_2\rangle$$

qQ eigenvalues

$(s = 1, \cdots, q,\ t = 1, \cdots, \mathcal{Q},\ n/q = \mathcal{N}/\mathcal{Q})$,
$\mathcal{N}$ and $\mathcal{Q}$: co-prime integers

qQ-fold degeneracy

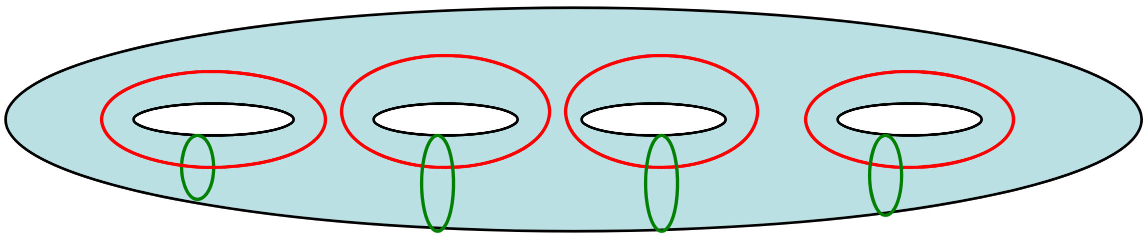
The minimal ground state degeneracy is the least common multiple of n and qQ=nQ^2/N

$(n/q = \mathcal{N}/\mathcal{Q}, \mathcal{N}, \mathcal{Q}$: co-prime integers)

$\Rightarrow$ $n\mathcal{Q}^2$-fold ground state degeneracy

In general

$\theta = \pi\frac{m}{n},\ e^* = \frac{p}{q}e$



$(n\mathcal{Q}^2)^g$-fold degeneracy

- Ground state degeneracy is obtained from charge fractionalization and fractional statistics
- It depends on the topology of the space

Test of our result
(torus)

$n\mathcal{Q}^2$-fold ground state degeneracy

$(e^* = ep/q, \theta = \pi m/n, n/q = \mathcal{N}/\mathcal{Q})$

1) Laughlin state (fractional Hall effect)

$e^* = e/q \qquad \theta = \pi/q$

D.Arovas, J.Schrieffer, F.Wilczek, '84

⇨ q-fold degeneracy

Agreement with numerical simulation and the Chern-Simon effective theory

W.P.Su '84, Wen-Niu '90

2) Kitaev model (topological computer, RVB state)

$e^* = e/2 \qquad \theta = 0$

⇨ 2^2-fold degeneracy

Agreement with the exact result

Kitaev '97

④ The physical meaning of λ

To consider the physical meaning of λ, we calculate the Hall conductance by using the linear response theory

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\underbrace{\int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y}_{\text{average over the boundary conditions}} \left[\langle \frac{\partial \phi_K}{\partial \theta_y} | \frac{\partial \phi_K}{\partial \theta_x} \rangle - (\theta_x \leftrightarrow \theta_y) \right]$$

Niu-Thouless-Wu '84

Here $\phi_K(\vec{\theta})$ is the ground state with the twisted boundary condition ($\vec{\theta} = (\theta_x, \theta_y)$, K degeneracy: $K = 1, \cdots, d$.)

$$\phi_K(x_i + L_1) = e^{i\theta_x}\phi_K(x_i)$$
$$\phi_K(y_i + L_2) = e^{i\theta_y}\phi_K(y_i)$$

The ground state satisfies

$$U_a|\phi_K(\vec{\theta})\rangle = e^{i\gamma_a(\vec{\theta})}|\phi_K(\vec{\theta} + 2\pi\widehat{e}_a)\rangle$$

$$\gamma_a(\vec{\theta}) = i\int_{\theta_a}^{\theta_a+2\pi}\langle\phi_K|\frac{\partial}{\partial\theta_a}|\phi_K\rangle$$

From $U_xU_y = e^{2\pi i\lambda}U_yU_x$

$$\gamma_x(\vec{\theta}+2\pi\widehat{e}_y)+\gamma_y(\vec{\theta}) = \gamma_y(\vec{\theta}+2\pi\widehat{e}_x)+\gamma_x(\vec{\theta})+2\pi\lambda+2\pi M$$

M: integer

Using this, we have

$$\begin{aligned}\sigma_{xy} &= \frac{e^2}{h}\int_0^{2\pi}\int_0^{2\pi}\frac{d\theta_x d\theta_y}{2\pi i}\left[\frac{\partial}{\partial\theta_y}\langle\phi_K|\frac{\partial}{\partial\theta_x}|\phi_K\rangle - (\theta_x \leftrightarrow \theta_y)\right]\\ &= -\frac{e^2}{h}\left[\int_0^{2\pi}\frac{d\theta_y}{2\pi}\frac{\partial\gamma_x(0,\theta_y)}{\partial\theta_y} - (\theta_x \leftrightarrow \theta_y)\right]\\ &= -\frac{e^2}{h}[\lambda + M]\end{aligned}$$

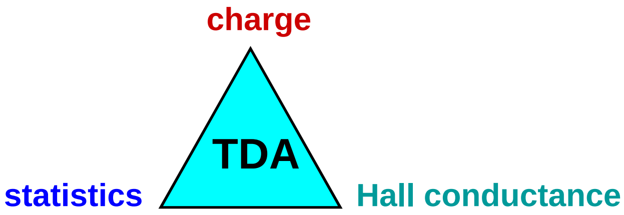
$\Rightarrow$

$$U_x U_y = e^{2\pi\sigma_{xy}h/e^2} U_y U_x$$

Fractional λ implies the fractional quantum Hall effect

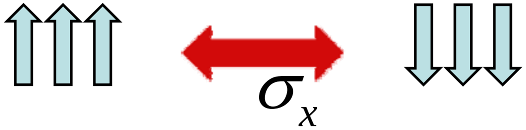
Part 1. Summary

◆ Using the braid group formulation, we found a closed algebraic structure which characterizes topological orders.



◆ Topological degeneracy is due to the topological discrete algebra.

cf.) For symmetry breaking orders



The degeneracy is due to broken generators.

◆The fractionalization implies non-trivial topological invariants

Part 2.

⑤ Quark deconfinement as a topological order

Idea

The Quark has **fractional charges**

↓

The deconfinement implies the topological order ?

yes

But.. the quark is an **elementary particle**, not a collective excitation.

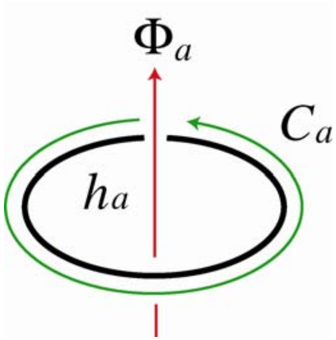
Nevertheless, non-trivial topological discrete algebra can be constructed in a similar manner ..

An important difference

$$T^3 = S^1 \otimes S^1 \otimes S^1$$

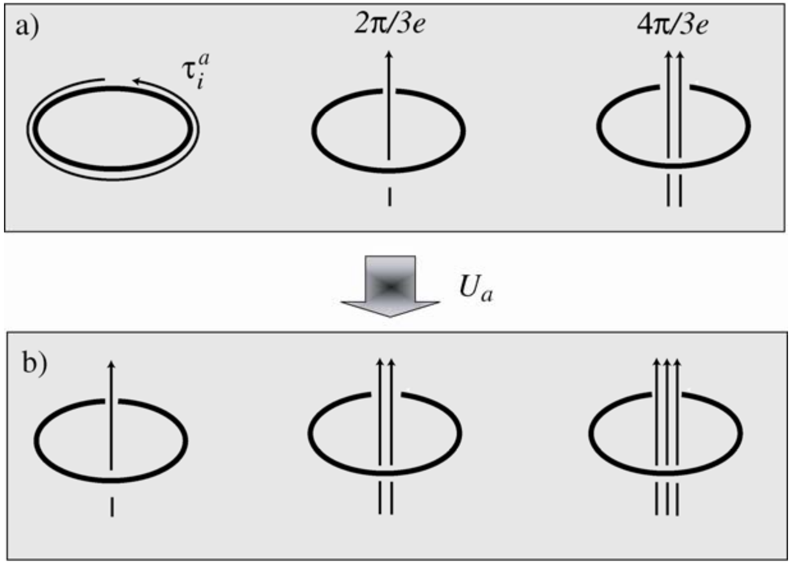
Due to the existence of gluons, the unit flux is reduced to $\Phi_a = 2\pi/3e$

e: the minimal charge of the constitute particle (charge of down quark)



cf.) For electron system, $\Phi_a = 2\pi/e$

Due to gluon fluctuations, there exist center vortices of SU(3) in each holes of three dimensional torus.



$$T^3 = S^1 \otimes S^1 \otimes S^1$$

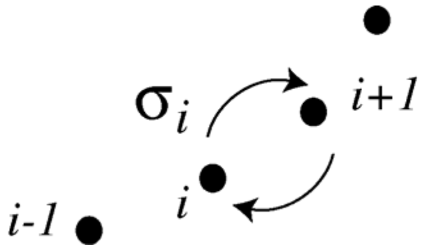
Thus, the physics is the same after the flux insertion by 2π/3e

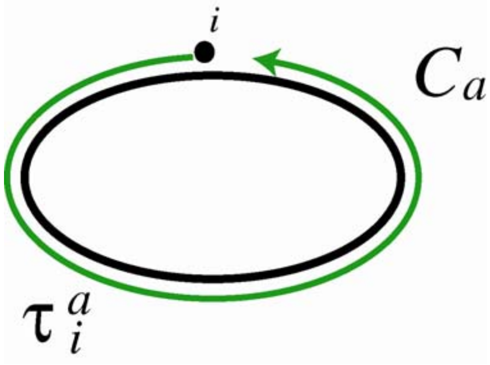
(not 2π/e)

The unit flux is reduced to $\Phi_a = 2\pi/3e$

permutation group on T^3

$$\sigma_k^2 = 1, \quad 1 \le k \le N-1,$$
$$(\sigma_k \sigma_{k+1})^3 = 1, \quad 1 \le k \le N-1$$
$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad 1 \le k \le N-3, \quad |l-k| \ge 2,$$
$$\tau_{i+1}^a = \sigma_i \tau_i^a \sigma_i, \quad 1 \le i \le N-1, \quad a = 1,2,3,$$
$$\tau_1^a \sigma_j = \sigma_j \tau_1^a, \quad 2 \le j \le N, \quad a = 1,2,3,$$
$$\tau_i^a \tau_j^b = \tau_j^b \tau_i^a, \quad i,j = 1, \cdots N, \quad a,b = 1,2,3.$$





The only allowed representation of the permutation group is boson or fermion. $\sigma_i = \pm 1$

The unique solution

$\Rightarrow$

$$\tau_i^a = T_a \text{ with } T_a T_b = T_b T_a$$

If quarks are **deconfined**, the physical states are classified with the permutation group of **quarks**.

⇓

Quarks have **fractional charges**, so we have non-trivial AB effects with inserted flux

⇓

Topological Discrete Algebra

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \qquad U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a$$

$$T_a T_b = T_b T_a$$

From the topological discrete algebra, the ground state degeneracy is found to be 3^3.

The quark deconfinement phase is topologically ordered !

On the other hand, …

If quarks are **confined**, the physical states are classified with the permutation group of **hadrons**.

⇓

Hadrons have **integer charges**, so we have no non-trivial AB effect with the inserted flux.

⇓

Topological discrete algebra is trivial

No topological degeneracy !

The confinement and deconfinement phases in QCD are discriminated by the topological ground state degeneracy !

For SU(N) QCD on $T^n \times R^{4-n}$

- deconfinement: N^n –fold ground state degeneracy
- confinement: No topological degeneracy

We check this in the following manners

- comparison with the heavy quark limit
- perturbative calculation of the topological ground state degeneracy
- consistency check with Fradkin-Shenker’s phase diagram
- comparison with Witten index

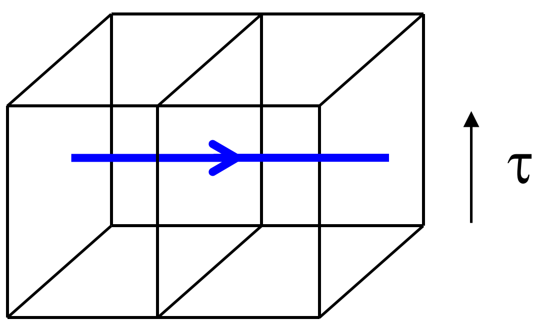
1. comparison with the heavy quark limit

QCD → SU(3) YM

heavy quark limit

The pure SU(3) YM has an additional symmetry known as center symmetry

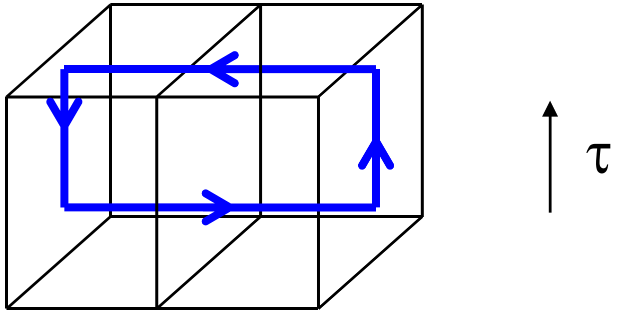
center symmetry



$$B_a : U_a \to e^{i2\pi/3} U_a$$

$$\begin{aligned} W(C_a) &\to B_a W(C_a) B_a^\dagger \\ &= e^{i2\pi/3} W(C_a) \end{aligned}$$

confinement phase



① area law

$$\langle W(C_a,\tau)W^\dagger(C_a,\tau')\rangle \sim e^{-\sigma L(\tau-\tau')}$$

temporal gauge

② cluster property

$$\langle W(C_a,\tau)W^\dagger(C_a,\tau')\rangle \overset{|\tau-\tau'|\to\infty}{\longrightarrow} |\langle W(C_a)\rangle|^2$$

⇒ $\langle W(C_a)\rangle = 0$

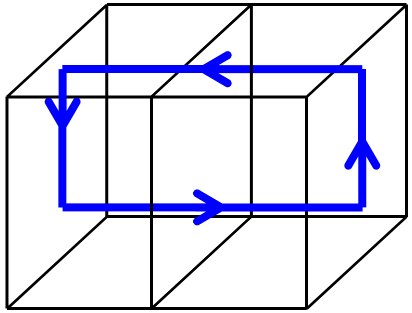
The center symmetry is not broken

⇒ **No ground state degeneracy**

deconfinement phase

① perimeter law

$$\langle W(C_a,\tau)W^\dagger(C_a,\tau'))\rangle \sim e^{-mL}$$



$\Longrightarrow$ $\langle W(C_a)\rangle \neq 0$

breaking of the center symmetry $\Longrightarrow$ **3^3 degeneracy**

The degeneracy reproduces the one obtained from the topological discrete algebra

In the static limit, our criterion for confinement coincides with the Wilson's.

remark

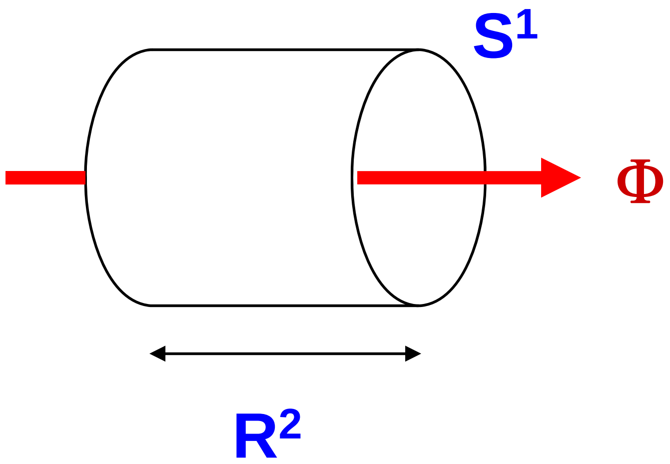
In this limit, topological discrete algebra becomes as follows.

$$\begin{cases} T_a \to W(C_a) \\ U_a \to B_a \\ U_a T_a = e^{i2\pi/3} T_a U_a \to B_a W(C_a) = e^{i2\pi/3} W(C_a) B_a \end{cases}$$

2. perturbative calculation of topological degeneracy

- We can calculate the topological degeneracy in the perturbation theory.
- Since the QCD is deconfined in the weak coupling, there must be the topological degeneracy.

SU(2) gauge theory on a $S^1 \times R^{1,2}$



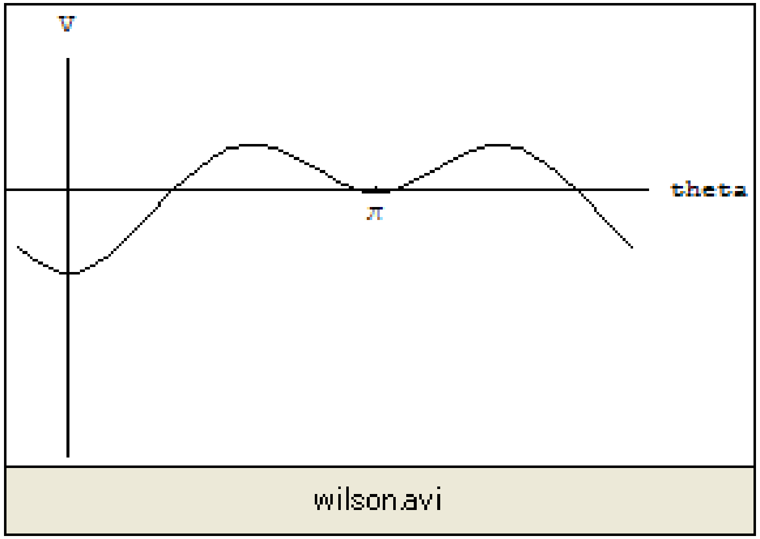
2-fold degeneracy should arises.

The Result

$$W(C) = P\,\text{exp}\left(ig\int_C \langle A_\mu(x)\rangle dx\right) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$V(\theta) = K\{-h(2\theta)+N_f[h(\theta+\Phi/2)-h(\theta-\Phi/2)]\}$$

N.Weiss 81,
Hosotani 83



Again, the result is consistent with the topological degeneracy

3. comparison with Fradkin-Shenker's phase diagram

Fradkin-Shenker's result (79)

- Higgs and the confinement phase are **smoothly connected** when the Higgs fields transform like **fundamental rep (complementarity)**.

- They are **separated by a phase boundary** when the Higgs fields transform like **other than fundamental rep**.

Our topological argument implies that **no ground state degeneracy** exists when Higgs and the confinement phase are **smoothly connected**.

4. N=1 SU(N) SYM

- Witten index has information on the ground state degeneracy on the torus T^3
- N=1 SU(N) SYM is in confinement phase, so there must be no topological degeneracy.

We know that there exists degeneracy on T^3

⇒ Witten index $\mathrm{Tr}(-1)^F = N$

↑

at least N-degeneracy

But …

- The theory has a discrete chiral symmetry, which is expected to be spontaneously broken.

$$\Psi \to e^{i\pi/2N}\Psi$$

- The rotation symmetry is expected not to broken.

$$\Psi \to -\Psi$$

N-fold degeneracy is a consequence of SSB of chiral symmetry

No topological degeneracy = quark confinement

◆ Summary of Part 2

◆ We can generalize the topological discrete algebra to the non-Abelian gauge theories.

◆ Topological degeneracy gives a gauge-invariant "order parameter" which distinguishes the quark confinement phase and deconfinement one.

$$\lim_{\beta\to\infty}\frac{\mathrm{Tr}_{\mathrm{T}^3}\left(e^{-\beta H}\right)}{\mathrm{Tr}_{\mathrm{S}^3}\left(e^{-\beta H}\right)}=1$$

◆ Topological criterion of the quark confinement is available even in the presence of the dynamical quarks.

◆ We can check our idea in various ways.

◆ Our topological discrete algebra can be generalized into other systems, (i.e. standard model, SUSY)